Generating Polynomials for Matrices with Toeplitz or Conjugate-Toeplitz Inverses

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ABSTRACT

A complex matrix $A=[a_{ij}]$ has been called conjugate-Toeplitz if $a_{ij}=c^{i-1}(a_{i-j})$, where $c(\ )$ denotes conjugation. A necessary and sufficient condition is derived for a matrix H to have a conjugate-Toeplitz inverse. The elements of H are generated from the coefficients of certain polynomials. The result is simplified either when H is to have a Toeplitz inverse, or when H is banded and is to have a conjugate-Toeplitz inverse. Each of these consequences of the main theorem is a different generalization of a previous result on banded matrices having a Toeplitz inverse.

1. INTRODUCTION

In earlier papers [2, 4] we defined an extension of Toeplitz matrices called conjugate-Toeplitz (CT) matrices, and showed that some known results for Toeplitz matrices could be generalized to the CT case.

It has been shown in [5] that certain banded matrices having Toeplitz inverses can be characterized by two polynomials. In this paper we demonstrate that the restriction to banded matrices is unnecessary. The results can also be extended to CT matrices, so that a matrix is the inverse of a CT matrix if and only if its elements are generated by the coefficients of two given polynomials. The proof used is different from that in [5]. In particular it reveals that the idea of two generating polynomials is closely related to formulae in [3] which also rely on two sets of parameters to define the inverse of a Toeplitz matrix. Indeed, since real Toeplitz matrices are a special case of CT matrices, all our results are stated and proved for the CT case, and the Toeplitz results then follow as corollaries.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

At present our main motivation for the introduction and study of CT matrices is the mathematically interesting fact that they lead to generalizations of many Toeplitz results. However, this revelation encourages the hope that CT matrices may be found useful in some of the many fields of applications (for example, control theory, signal processing, and statistics), where Toeplitz-type matrices have been employed.

It is convenient at this point to record some basic definitions. The complex conjugate $\bar{x}$ of a complex number x is denoted by $c(x)$, so that $c^{2m}(x)=x$ and $c^{2m-1}(x)=\bar{x}$ for positive integers m. If $A=[a_{ij}]$, then we define $c^r(A)=[c^r(a_{ij})]$.

An $n\times n$ matrix A over $\mathbf{C}$ is *conjugate-Toeplitz* (CT) if

$$a_{ij}=c^{i-1}(a_{i-j}),\qquad i,j=1,2,\dots,n. \tag{1.1}$$

If a_{ij} are all real, $a_{ij}=a_{i-j}$, which defines a Toeplitz matrix. We also need to use the reverse unit matrix of order n, $J_n=[\delta_{i,n-j+1}]$, where δ_{ij} is the Kronecker delta.

The organization of the paper is as follows. In Section 2 we state and prove some preliminary lemmas which are needed for the proof of the main result (Theorem 3.1), which is given in Section 3. This gives the necessary and sufficient conditions for a given matrix to have an inverse in CT form. An illustrative numerical example is presented in Section 4. In Section 5 two special cases of Theorem 3.1 are derived. The first (Theorem 5.1) gives the necessary and sufficient conditions for a matrix to have a Toeplitz inverse. The result in [5] on banded matrices is then seen to be a special case of this. Moreover, this result in [5] is extended in a different way in Theorem 5.2 to characterize when a banded matrix has a CT inverse.

2. PRELIMINARY LEMMAS

Before considering the main theorem of this paper, we need to define a matrix which plays an important role in this theorem and in some of the lemmas to be used in the proof.

Definition 2.1. Given two polynomials of degree $n-1$

$$A(x)=\sum_{\mu=0}^{n-1}\alpha_\mu x^\mu,\qquad B(x)=\sum_{\mu=0}^{n-1}\beta_\mu x^\mu, \tag{2.1}$$

let a $(2n-2)$th-order matrix S associated with $A(x)$ and $B(x)$ be defined as

$$S=\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$$

$$=\left[\begin{array}{ccc|ccc} c(\alpha_0) & \cdots & c^{n-1}(\alpha_{n-2}) & c^n(\alpha_{n-1}) & & 0 \\ & \ddots & \vdots & \vdots & \ddots & \\ 0 & & c^{n-1}(\alpha_0) & c^n(\alpha_1) & \cdots & c^{2n-2}(\alpha_{n-1}) \\ \hline c(\beta_{n-1}) & \cdots & c^{n-1}(\beta_1) & c^n(\beta_0) & & 0 \\ & \ddots & \vdots & \vdots & \ddots & \\ 0 & & c^{n-1}(\beta_{n-1}) & c^n(\beta_{n-2}) & \cdots & c^{2n-2}(\beta_0) \end{array}\right] \tag{2.2}$$

In all that follows it will be assumed that $\alpha_0 \neq 0$ and $\beta_0 = 1$. It will be seen later that S is closely related to the Sylvester matrix defined in (5.1), details of which can be found in [1].

LEMMA 2.1. *Two polynomials*

$$C(x)=\sum_{\mu=0}^{n-1}\gamma_\mu x^\mu, \qquad D(x)=\sum_{\mu=0}^{n-1}\delta_\mu x^\mu, \tag{2.3}$$

related to $A(x)$ and $B(x)$ in (2.1) by the equations

$$\sum_{k=0}^{n-i}\left[c^{i-1}(\alpha_k)\delta_{k+i-1}-c^{i-1}(\beta_{k+i-1})\gamma_k\right]=0, \qquad i=2,3,\dots,n, \tag{2.4}$$

$$\sum_{k=0}^{n-i}\left[c^{i-1}(\alpha_{k+i-1})\delta_k-c^{i-1}(\beta_k)\gamma_{k+i-1}\right]=0, \qquad i=1,2,\dots,n, \tag{2.5}$$

with $\delta_0=1$, are uniquely defined by (2.4) and (2.5) if and only if S is nonsingular.

Proof. If we write the equations (2.4), $i=n,n-1,\dots,2$ and (2.5), $i=1,2,\dots,n$, in that order, as a set of linear equations in δ_μ, $\mu=n-1$,

$n-2,\dots,1$, and γ_μ, $\mu=0,1,\dots,n-1$, and compare with (2.2), we obtain

$$\begin{bmatrix} c^n(S_1^T) & -c^n(S_3^T) & \phi \\ c^n(S_2^T) & -c^n(S_4^T) & z \\ \phi^T & \phi^T & -c^{n-1}(\beta_0) \end{bmatrix} \begin{bmatrix} \delta_{n-1} \\ \vdots \\ \delta_1 \\ \hline \gamma_0 \\ \vdots \\ \gamma_{n-1} \end{bmatrix} = \begin{bmatrix} \phi \\ \hline -\alpha_0 \\ \vdots \\ -c^{n-1}(\alpha_{n-1}) \end{bmatrix}, \tag{2.6}$$

where ϕ denotes the $(n-1)\times 1$ zero matrix and $z=[-\beta_{n-1},\dots,-c^{n-2}(\beta_1)]^T$. Since $\beta_0=1$, the determinant of the first matrix in (2.6) is

$$-\begin{vmatrix} c^n(S_1^T) & -c^n(S_3^T) \\ c^n(S_2^T) & -c^n(S_4^T) \end{vmatrix} = -\left| c^n \begin{bmatrix} S_1^T & S_3^T \\ S_2^T & S_4^T \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ 0 & -I_{n-1} \end{bmatrix} \right|$$

$$= (-1)^n c^n |S^T|, \tag{2.7}$$

where I_{n-1} is the unit matrix of order $n-1$. Hence this determinant is nonzero if and only if S is nonsingular, and this is therefore the condition for the solution of (2.6) to be unique. ∎

LEMMA 2.2. *If two nth order matrices H and H_1 are defined by*

$$H = \begin{bmatrix} \delta_0 & & & \\ c(\delta_1) & c(\delta_0) & 0 & \\ \vdots & & \ddots & \\ c^{n-1}(\delta_{n-1}) & \cdots & c^{n-1}(\delta_1) & c^{n-1}(\delta_0) \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_0 & c(\alpha_1) & \cdots & c^{n-1}(\alpha_{n-1}) \\ & c(\alpha_0) & \cdots & c^{n-1}(\alpha_{n-2}) \\ & 0 & \ddots & \vdots \\ & & & c^{n-1}(\alpha_0) \end{bmatrix}$$

$$
-\begin{bmatrix} 0 & & & \\ c(\gamma_{n-1}) & 0 & \mathbf{0} & \\ \vdots & & \ddots & \\ c^{n-1}(\gamma_1) & \cdots & c^{n-1}(\gamma_{n-1}) & 0 \end{bmatrix}
\times \begin{bmatrix} 0 & c(\beta_{n-1}) & \cdots & c^{n-1}(\beta_1) \\ & & \ddots & \vdots \\ & \mathbf{0} & & c^{n-1}(\beta_{n-1}) \\ & & & 0 \end{bmatrix} \tag{2.8}
$$

$$
H_1 = \begin{bmatrix} \beta_0 & & & \\ c(\beta_1) & c(\beta_0) & & \mathbf{0} \\ \vdots & & \ddots & \\ c^{n-1}(\beta_{n-1}) & \cdots & c^{n-1}(\beta_1) & c^{n-1}(\beta_0) \end{bmatrix}
\times \begin{bmatrix} \gamma_0 & c(\gamma_1) & \cdots & c^{n-1}(\gamma_{n-1}) \\ & c(\gamma_0) & \cdots & c^{n-1}(\gamma_{n-2}) \\ & \mathbf{0} & \ddots & \vdots \\ & & & c^{n-1}(\gamma_0) \end{bmatrix}
$$

$$
-\begin{bmatrix} 0 & & & \\ c(\alpha_{n-1}) & & \mathbf{0} & \\ \vdots & \ddots & & \\ c^{n-1}(\alpha_1) & \cdots & c^{n-1}(\alpha_{n-1}) & 0 \end{bmatrix}
\times \begin{bmatrix} 0 & c(\delta_{n-1}) & \cdots & c^{n-1}(\delta_1) \\ & & \ddots & \vdots \\ & \mathbf{0} & & c^{n-1}(\delta_{n-1}) \\ & & & 0 \end{bmatrix} \tag{2.9}
$$

where $\beta_0 = \delta_0 = 1$ *and* $\alpha_0 \neq 0$, *then* $H_1 = c^{n+1}(JH^TJ)$ *if and only if equations* (2.4) *and* (2.5) *are satisfied.*

Proof. Suppose $H_1 = c^{n+1}(JH^TJ)$. If we conjugate this equation $n+1$ times and note that $J^2 = I$, we obtain $H = c^{n+1}(JH_1^TJ)$. Hence, using the expression for H_1 from (2.9),

$$H = \begin{bmatrix} \gamma_{n-1} & \cdots & \gamma_0 \\ \vdots & \iddots & \\ c^{n-1}(\gamma_0) & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \beta_{n-1} & \cdots & c^{n-1}(\beta_0) \\ \vdots & \iddots & \\ \beta_0 & & \mathbf{0} \end{bmatrix}$$
$$- \begin{bmatrix} \delta_1 & \cdots & \delta_{n-1} & 0 \\ \vdots & \iddots & & \\ c^{n-2}(\delta_{n-1}) & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \alpha_1 & \cdots & c^{n-2}(\alpha_{n-1}) & 0 \\ \vdots & \iddots & & \\ \alpha_{n-1} & & \mathbf{0} & \\ 0 & & & \end{bmatrix}. \tag{2.10}$$

Comparing the i, j elements in (2.10) and (2.8), we obtain

$$\sum_{k=0}^{\min(i-1,j-1)} c^{i-1}(\delta_{i-k-1})c^{j-1}(\alpha_{j-k-1})$$
$$- \sum_{l=1}^{\min(i-1,j-1)} c^{i-1}(\gamma_{n+l-i})c^{j-1}(\beta_{n+l-j})$$
$$= \sum_{m=0}^{\min(n-i,n-j)} c^{i-1}(\gamma_{n-i-m})c^{j-1}(\beta_{n-j-m})$$
$$- \sum_{p=1}^{\min(n-i,n-j)} c^{i-1}(\delta_{i+p-1})c^{j-1}(\alpha_{j+p-1})$$
$$i, j = 1,2,\dots,n. \tag{2.11}$$

If in the third sum in (2.11) m is replaced by $-l$ and in the fourth sum p is replaced by $-k$, Equation (2.11) becomes

$$\sum_{k=\max(-n+i,\,-n+j)}^{\min(i-1,j-1)} c^{i-1}(\delta_{i-k-1})c^{j-1}(\alpha_{j-k-1})$$
$$- \sum_{l=\max(-n+i,\,-n+j)}^{\min(i-1,j-1)} c^{i-1}(\gamma_{n+l-i})c^{j-1}(\beta_{n+l-j}) = 0. \tag{2.12}$$

If $i > j$, let $K = j - k - 1$, $L = n + l - i$. Then (2.12) becomes

$$\sum_{K=0}^{n-i+j-1} c^{i-1}(\delta_{K-j+i})c^{j-1}(\alpha_K) - \sum_{L=0}^{n-i+j-1} c^{i-1}(\gamma_L)c^{j-1}(\beta_{L-j+i}) = 0 \tag{2.13}$$

Finally, if we let $i - j + 1 = q$ in (2.13), we have $q = 2, 3, \ldots, n$ and

$$\sum_{K=0}^{n-q} c^{q+j-2}(\delta_{K+q-1})c^{j-1}(\alpha_K) - \sum_{L=0}^{n-q} c^{q+j-2}(\gamma_L)c^{j-1}(\beta_{L+q-1}) = 0. \tag{2.14}$$

Conjugating (2.14) $q + j$ times gives (2.4).

A similar analysis for the case $i \leqslant j$ gives (2.5).

Conversely, suppose (2.4) and (2.5) are given. Then reversing the above argument shows that if H and H_1 are given by (2.8) and (2.9) respectively, then $H_1 = c^{n+1}(JH^TJ)$. ■

Conditions on α_0, β_0, and δ_0 were imposed in the statement of Lemma 2.2, and we can now show that if in addition S is nonsingular, then γ_0 is nonzero.

Lemma 2.3. *If $C(x)$ is defined as in Lemma 2.1 and S in (2.2) is nonsingular, then*

$$\gamma_0 = c^{n-1}\left[\alpha_0 \frac{|S|}{|\bar{S}|}\right]. \tag{2.15}$$

Proof. Using Cramer's rule to solve (2.6) for γ_0 and expanding the numerator by the upper left element and the denominator by the lower right element gives

$$\gamma_0 = -c^{n-1}\left[\alpha_0 \frac{\begin{vmatrix} S_1^T L & -S_3^T \\ S_2^T L & -S_4^T \end{vmatrix}}{\begin{vmatrix} \bar{S}_1^T & -\bar{S}_3^T \\ \bar{S}_2^T & -\bar{S}_4^T \end{vmatrix}}\right]$$

$$= -c^{n-1}\left[\alpha_0 \frac{|S^T|\begin{vmatrix} L & 0 \\ 0 & -I_{n-1} \end{vmatrix}}{|\bar{S}^T|\begin{vmatrix} I_{n-1} & 0 \\ 0 & -I_{n-1} \end{vmatrix}}\right], \tag{2.16}$$

where S_r, $r = 1,2,3,4$, are defined in (2.2) and L is the matrix of order $n-1$

$$L = \begin{bmatrix} I_{n-2} & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.17}$$

A simplification of (2.16) now gives (2.15). ∎

Finally we need the following result, which is a generalization of Theorem 6.1, p. 86 in [3]. This generalization is proved in [2].

LEMMA 2.4. *Let $A = [a_{ij}]$ be an arbitrary nonsingular CT matrix of order n such that $a_{ij} = c^{i-1}[a_{i-j}]$, and let $B = c^{n+1}(JA^TJ)$, so that $b_{ij} = c^{j-i}(a_{i-j})$. Then if the $(1,1)$ element of A^{-1} is nonzero, A^{-1} and B^{-1} can be written in the form (2.8) and (2.9) where (letting $\beta_0 = \delta_0 = 1$ and $I = [\delta_{ij}]$) α_μ, β_μ, γ_μ, and δ_μ, $\mu = 0,1,\dots,n-1$, satisfy*

$$\alpha_0 \sum_{s=1}^{n} c^{i-1}(a_{i-s})c^{s-1}(\delta_{s-1}) = \delta_{i1}, \qquad i = 1,2,\dots,n, \tag{2.18}$$

$$\sum_{s=1}^{n} c^{s-1}(a_{s-j})c^{s-1}(\alpha_{s-1}) = \delta_{1j}, \qquad j = 1,2,\dots,n, \tag{2.19}$$

$$\gamma_0 \sum_{s=1}^{n} c^{s-1}(a_{i-s})c^{s-1}(\beta_{s-1}) = \delta_{i1}, \qquad i = 1,2,\dots,n, \tag{2.20}$$

$$\sum_{s=1}^{n} c^{j-1}(a_{s-j})c^{s-1}(\gamma_{s-1}) = \delta_{1j}, \qquad j = 1,2,\dots,n. \tag{2.21}$$

3. THE CHARACTERIZATION OF MATRICES WITH CT INVERSES

For convenience of reference to the previous section, the same alphabetical symbols will be used in the main theorem as in the lemmas.

THEOREM 3.1. *A matrix $H = [h_{ij}]$, $i, j = 1,2,\dots,n$, of order n whose $(1,1)$ element is nonzero is the inverse of a CT matrix if and only if h_{ij} is the coefficient of x^j in*

$$x^i\left\{c^{j-1}[A(x)]\,c^{i-1}\left[\sum_{\mu=0}^{i-1}\delta_\mu x^{-\mu}\right] - c^{j-1}\left[B\left(\frac{1}{x}\right)\right]c^{i-1}\left[\sum_{\mu=n-i+1}^{n-1}\gamma_\mu x^\mu\right]\right\}, \tag{3.1}$$

where

$$A(x)=\sum_{\mu=0}^{n-1}\alpha_\mu x^\mu,\quad B(x)=\sum_{\mu=0}^{n-1}\beta_\mu x^\mu,\qquad \alpha_0\neq 0,\quad \beta_0=1,\tag{3.2}$$

and the coefficients in (3.2) *satisfy the condition that S as defined in* (2.2) *is nonsingular. (The coefficients* γ_μ *and* δ_μ, $\mu=0,1,\dots,n-1$ *are the unique solutions of* (2.4) *and* (2.5), *and* Σ_n^{n-1} *is defined as zero.)*

Before the proof is shown it should be pointed out that (3.1) and (2.8) are equivalent, since both can be obtained from (3.3) below. The theorem is stated in the above form in order to compare it with the corresponding theorem in [5].

Proof. Firstly suppose that H is the inverse of a CT matrix. We can then set $A^{-1}=H$ in Lemma 2.4, so that H can be written in the form (2.8). Since H is nonsingular, its (1,1) element (α_0) is nonzero. Taking the particular choice $\beta_n=\gamma_n=0$ we obtain

$$h_{ij}=\sum_{k=0}^{\min(i-1,j-1)}\Big[c^{i-1}(\delta_{i-k-1})c^{j-1}(\alpha_{j-k-1}) - c^{i-1}(\gamma_{n-i+1+k})c^{j-1}(\beta_{n-j+1+k})\Big],\qquad i,j=1,2,\dots,n.\tag{3.3}$$

It can easily be verified that (3.3) is the coefficient of x^j in (3.1). Next, Lemma 2.4 implies that there exists a matrix $H_1=B^{-1}$ such that $H_1=c^{n+1}(JH^TJ)$. Hence Lemma 2.2 shows that (2.4) and (2.5) are satisfied, these being used to define γ_μ and δ_μ. It therefore remains to prove that $|S|\neq 0$. In order to do this, introduce the partitioned matrix

$$T=\begin{bmatrix}T_1 & -T_2\\ 0 & I_{n-1}\end{bmatrix},\tag{3.4}$$

where T_1 and T_2 are the lower triangular CT matrices of order $n-1$

$$T_1=\begin{bmatrix}c(\delta_0) & & 0\\ \vdots & \ddots & \\ c^{n-1}(\delta_{n-2}) & \cdots & c^{n-1}(\delta_0)\end{bmatrix},$$

$$T_2=\begin{bmatrix}c(\gamma_{n-1}) & & 0\\ \vdots & \ddots & \\ c^{n-1}(\gamma_1) & \cdots & c^{n-1}(\gamma_{n-1})\end{bmatrix}.\tag{3.5}$$

Then from (2.2) and (3.4)

$$TS = \begin{bmatrix} T_1S_1 - T_2S_3 & T_1S_2 - T_2S_4 \\ S_3 & S_4 \end{bmatrix}, \tag{3.6}$$

and in fact

$$T_1S_2 - T_2S_4 = 0. \tag{3.7}$$

To show this, note that since T_1, T_2, S_2, and S_4 are lower triangular CT matrices, it is easy to verify that $T_1S_2 - T_2S_4$ is also a lower triangular CT matrix. Hence it is only necessary to prove that the elements in the first column of $T_1S_2 - T_2S_4$ are zero. We have

$$[T_1S_2 - T_2S_4]_{i1} = \sum_{k=0}^{i-1} \left[c^i(\delta_k)c^n(\alpha_{n-i+k}) - c^n(\beta_k)c^i(\gamma_{n-i+k})\right],$$

$$i = 1,2,\ldots,n-1. \tag{3.8}$$

If we put $j = n - i + 1$ in the right hand side of (3.8) and conjugate $n + j - 1$ times, we obtain (2.5), which confirms (3.7). Hence from (3.6) we have

$$|TS| = |T_1S_1 - T_2S_3||S_4| \tag{3.9}$$

and since $|T| = |S_4| = 1$, it follows that $|S| \neq 0$ if and only if $|T_1S_1 - T_2S_3| \neq 0$. In order to determine $|T_1S_1 - T_2S_3|$, consider H in the form (2.8). If

$$P = \begin{bmatrix} 1 & 0 \\ -c(\delta_1) & \\ \vdots & I_{n-1} \\ -c^{n-1}(\delta_{n-1}) & \end{bmatrix}, \tag{3.10}$$

then premultiplying (2.8) by P and taking determinants shows that

$$|P||H| = \alpha_0|T_1S_1 - T_2S_3| \tag{3.11}$$

Equations (3.9) and (3.11) show that $|S| \neq 0$, since H is nonsingular, which establishes the first part of the theorem.

Conversely, suppose that (3.1) holds, and (3.2) is given such that S defined by (2.2) is nonsingular, and that γ_μ and δ_μ, $\mu = 0, 1, \dots, n-1$, are the unique solutions of (2.4) and (2.5) by Lemma 2.1. It is to be proved that H^{-1} is a CT matrix.

It is easy to verify that if (3.1) holds, then H can be written in the form (2.8). The next step is to show that, given α_μ and β_μ, there exists a unique set a_r, $r = -n+1, -n+2, \dots, n-1$, satisfying (2.19) and (2.20). If we write (2.19), $j = n-1, \dots, 1, n$, and (2.20), $i = 1, 2, \dots, n$, in matrix form, we have

$$\begin{bmatrix} \phi & c^{n-1}(S_1) & c^{n-1}(S_2) \\ c^{n-1}(\alpha_0) & x^T & \phi^T \\ c^{n-1}(\beta_{n-1}) & y^T & \phi^T \\ \phi & c^{n-1}(S_3) & c^{n-1}(S_4) \end{bmatrix} \begin{bmatrix} c^{n-1}(a_{-n+1}) \\ c^{n-2}(a_{-n+2}) \\ \vdots \\ c^{n-1}(a_{n-1}) \end{bmatrix} = \begin{bmatrix} e_{n-1} \\ 0 \\ \gamma_0^{-1} \\ \phi \end{bmatrix}, \tag{3.12}$$

where $x^T = [c^{n-2}(\alpha_1), \dots, \alpha_{n-1}]$, $y^T = [c^{n-2}(\beta_{n-2}), \dots, \beta_0]$, and e_{n-1} is the last column of I_{n-1}. Note that $\gamma_0 \neq 0$, by Lemma 2.3. If we omit the $(n+1)$th equation from (3.12), the coefficient matrix of the reduced system has a determinant which is nonzero, since $\alpha_0 \neq 0$ and S is nonsingular. If we denote the coefficient matrix on the left hand side of (3.12) by N, then N has rank $2n-1$, the maximum possible. Hence for consistency of the equations (3.12), the augmented matrix, namely

$$M = \begin{bmatrix} N & \begin{matrix} e_{n-1} \\ 0 \\ \gamma_0^{-1} \\ \phi \end{matrix} \end{bmatrix},$$

must also have rank $2n-1$, and must therefore be singular.

Now expand $|M|$ by the last column to obtain, after some manipulation,

$$|M| = \gamma_0^{-1} c^{n-1}(\alpha_0) c^{n-1}|S| - c^{n-1}(\beta_0) c^n |S| \tag{3.13}$$

Since $\beta_0 = 1$, (3.13) is zero by Lemma 2.3. Thus the uniqueness of the solution of (3.12), i.e. of (2.19) and (2.20), is established.

Next, consider (2.19) and (2.20) as linear equations in α_i and β_i, $i = 0, 1, \dots, n-1$. (Recall that these equations are consistent.) They can be written in the form

$$A^T\begin{bmatrix}\alpha_0\\ c(\alpha_1)\\ \vdots\\ c^{n-1}(\alpha_{n-1})\end{bmatrix}=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}, \tag{3.14}$$

where A is the CT matrix in Lemma 2.4, and

$$c^{n+1}(JA^TJ)\begin{bmatrix}\beta_0\\ c(\beta_1)\\ \vdots\\ c^{n-1}(\beta_{n-1})\end{bmatrix}=\begin{bmatrix}\frac{1}{\gamma_0}\\0\\ \vdots\\0\end{bmatrix}. \tag{3.15}$$

Conjugating (3.15) $n+1$ times and premultiplying by J gives

$$A^T\begin{bmatrix}\beta_{n-1}\\ c(\beta_{n-2})\\ \vdots\\ c^{n-1}(\beta_0)\end{bmatrix}=\begin{bmatrix}0\\0\\ \vdots\\ \dfrac{1}{c^{n-1}(\gamma_0)}\end{bmatrix}. \tag{3.16}$$

To show that A is nonsingular, assume that A^T is singular. Then the rows $(\rho_1,\dots,\rho_n)$ of A^T are linearly dependent. Equation (3.14) implies that $\rho_1\neq 0$ and hence one of ρ_i is expressible in terms of those preceding it. Let p $(\geqslant 2)$ be the largest number such that

$$\rho_p=\sum_{k=1}^{p-1} w_k\rho_k \tag{3.17}$$

Considering the $(s-1)$th element of ρ_p for $s=2,3,\dots,n$, (3.17) gives

$$c^{s-2}(a_{-p+s-1})=\sum_{k=1}^{p-1} w_k c^{s-2}(a_{-k+s-1}),$$

which can be rewritten as

$$c^{s-2}(a_{-p-1+s}) = \sum_{k=2}^{p} w_{k-1} c^{s-2}(a_{s-k}), \qquad s = 2,3,\ldots,n \tag{3.18}$$

Next, we conjugate (3.18), multiply it by $c^{s-1}(\alpha_{s-1})$ and sum for s from 2 to n to obtain

$$\sum_{s=2}^{n} c^{s-1}(a_{-p-1+s}) c^{s-1}(\alpha_{s-1}) = \sum_{k=2}^{p} c(w_{k-1}) \sum_{s=2}^{n} c^{s-1}(a_{s-k}) c^{s-1}(\alpha_{s-1}). \tag{3.19}$$

Equation (2.19) shows that

$$\sum_{s=2}^{n} c^{s-1}(a_{s-k}) c^{s-1}(\alpha_{s-1}) = -a_{1-k}\alpha_0, \tag{3.20}$$

and if $p < n$,

$$\sum_{s=2}^{n} c^{s-1}(a_{-p-1+s}) c^{s-1}(\alpha_{s-1}) = -a_{-p}\alpha_0. \tag{3.21}$$

Since $\alpha_0 \neq 0$, substituting (3.21) and (3.20) into (3.19) gives

$$a_{-p} = \sum_{k=2}^{p} c(w_{k-1}) a_{1-k} \tag{3.22}$$

Finally if (3.18) is conjugated and combined with (3.22) we have

$$c^{s-1}(a_{-p-1+s}) = \sum_{k=2}^{p} c(w_{k-1}) c^{s-1}(a_{s-k}), \qquad s = 1,2,\ldots,n,$$

which is the same as

$$\rho_{p+1} = \sum_{k=2}^{p} c(w_{k-1}) \rho_k.$$

This contradicts the choice of p as the largest number to satisfy (3.17). Hence $p = n$, and so ρ_n can be expressed as a linear combination of the preceding rows. This however contradicts (3.16), and so A^T and therefore A is nonsingular.

This part of the proof is similar to that on p. 87 in [3]. We now have a CT matrix $A = [a_{ij}]$ with $a_{1j} = a_{1-j}$, $j = 1,2,\dots,n$, and $a_{i1} = c^{i-1}(a_{i-1})$, $i = 2,3,\dots,n$, such that its inverse has the form (2.8) for *some* γ_μ, δ_μ, $\mu = 0,1,\dots,n-1$. There will then also exist a related matrix $B = c^{n+1}(JA^TJ)$ such that B^{-1} can be written in the form (2.9), and (2.18) and (2.21) will be satisfied by Lemma 2.4. But by Lemma 2.2, the relationship between A and B implies that γ_μ and δ_μ are given by (2.4) and (2.5), which by assumption are prespecified, and hence $H = A^{-1}$. ■

4. NUMERICAL EXAMPLE

To illustrate Theorem 3.1, arbitrarily select in (3.2) the two polynomials of degree three

$$A(x) = (1+i) + (2-i)x + 2x^2 - ix^3 \tag{4.1}$$

and

$$B(x) = 1 + (1-i)x + (3+i)x^2 - x^3. \tag{4.2}$$

Then defining S by (2.2) gives $|S| = 87 - 13i$, so that S is nonsingular. If we now use (2.6) to obtain γ_μ and δ_μ, $\mu = 0,1,2,3$, with $\delta_0 = 1$, we obtain

$$\begin{aligned} C(x) = \tfrac{1}{3869}\big[&(4831 - 2569i) + (5159 - 3276i)x \\ &+ (2105 - 397i)x^2 + 3869ix^3\big] \end{aligned} \tag{4.3}$$

$$\begin{aligned} D(x) = \tfrac{1}{3869}\big[&3869 + (-3472 - 1764i)x \\ &+ (9698 - 10736i)x^2 + (-3700 - 1131i)x^3\big] \end{aligned} \tag{4.4}$$

and so from (2.8) or (3.1) we can obtain h_{ij} and

$$H = \frac{1}{3869}\begin{bmatrix} 3869+3869i & 7738+3869i & 7738 & 3869i \\ -5236-1708i & -4839-7682i & -3075+11266i & 2105+397i \\ 20434-1038i & 27001-10463i & 7845-14704i & 5159-3276i \\ -4831-2569i & 17062+2876i & 7400-2262i & 4831+2569i \end{bmatrix}. \tag{4.5}$$

This matrix H is the inverse of a CT matrix A, which we will now construct. To determine A we use (3.12), omitting the fifth equation, and obtain

$$A=\frac{1}{7738}\begin{bmatrix} 57-809i & 609+91i & 935+305i & -1574+1010i \\ -49+2053i & 57+809i & 609-91i & 935-305i \\ 1731+275i & -49-2053i & 57-809i & 609+91i \\ -3855-5561i & 1731-275i & -49+2053i & 57+809i \end{bmatrix}. \tag{4.6}$$

One can check that $HA = I$.

REMARK 1. The construction of H can be simplified by using (3.1) together with (2.4) and (2.5), once all four polynomials $A(x)$, $B(x)$, $C(x)$, and $D(x)$ are determined. This will be illustrated for a fourth order matrix H. From (3.1) and using (2.4) and (2.5) we obtain

$$H=\begin{bmatrix} \alpha_0 & \bar{\alpha}_1 & \alpha_2 & \bar{\alpha}_3 \\ \alpha_0\bar{\delta}_1 & \bar{\alpha}_1\bar{\delta}_1+\bar{\alpha}_0-\bar{\beta}_3\bar{\gamma}_3 & \alpha_2\bar{\delta}_1+\alpha_1-\beta_2\bar{\gamma}_3 & \bar{\gamma}_2 \\ \alpha_0\delta_2 & \bar{\alpha}_1\delta_2+\bar{\alpha}_0\delta_1-\bar{\beta}_3\gamma_2 & \beta_1\gamma_1+\gamma_0-\alpha_3\delta_3 & \gamma_1 \\ \alpha_0\bar{\delta}_3 & \bar{\beta}_2\bar{\gamma}_0 & \beta_1\bar{\gamma}_0 & \bar{\gamma}_0 \end{bmatrix} \tag{4.7}$$

Thus for $i + j \leqslant n + 1$, (3.1) is used directly, and for $i + j > n + 1$ it is used in conjunction with (2.4) and (2.5).

This simplification occurs because $c^{n+1}(JH^TJ) = B^{-1}$, which, since B is CT, will also satisfy Theorem 3.1, with the roles of $A(x)$ and $B(x)$ and those of $C(x)$ and $D(x)$ being reversed. Hence, for example, the first row of B^{-1} is $[\gamma_0, \bar{\gamma}_1, \gamma_2, \bar{\gamma}_3]$, which is the conjugate of the last column of H with the elements reversed. Note that $\gamma_3 = \bar{\alpha}_3$ from (2.6).

5. CONSEQUENCES OF THEOREM 3.1

5.1. *Toeplitz Matrices*

It is clear that if in Lemma 2.4 and Theorem 3.1 the elements of the matrices are restricted to $\mathbb{R}$, then results are obtained for real Toeplitz matrices. Because Toeplitz matrices are persymmetric, A and B in Lemma 2.4 are identical and so (2.18) and (2.21) are redundant. Also, by equating the first rows and first columns of (2.8) and (2.9), one sees that $A(x) \equiv C(x)$ and

$B(x) \equiv D(x)$, and therefore Equations (2.4) and (2.5) are also redundant. If however the proof is reexamined, it becomes clear that the simplification comes from the persymmetry of a Toeplitz matrix and it is not in fact necessary to restrict the elements to be real. It should also be pointed out that the proof is very much simpler. Lemmas 2.1, 2.2, and 2.3 are no longer required, and S as defined in (2.2) is replaced by the Sylvester matrix

$$\hat{S} = \begin{bmatrix} \hat{S}_1 & \hat{S}_2 \\ \hat{S}_3 & \hat{S}_4 \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc} \alpha_0 & \cdots & & \alpha_{n-2} & \alpha_{n-1} & & & 0 \\ & \ddots & & \vdots & \vdots & & & \\ 0 & & & \alpha_0 & \alpha_1 & & \cdots & \alpha_{n-1} \\ \hline \beta_{n-1} & \cdots & & \beta_1 & \beta_0 & & & 0 \\ & \ddots & & \vdots & \vdots & \ddots & & \\ 0 & & & \beta_{n-1} & \beta_{n-2} & & \cdots & \beta_0 \end{array}\right]. \tag{5.1}$$

Hence in the first part of the proof in Theorem 3.1, concerning the nonsingularity of S, when H is assumed to be the inverse of a Toeplitz matrix, it follows that S is replaced by $\hat{S}$, and T in (3.4) is replaced by

$$\hat{T} = \begin{bmatrix} \hat{S}_4 & -\hat{S}_2 \\ 0 & I_{n-1} \end{bmatrix}. \tag{5.2}$$

Thus (3.6) becomes, using (5.1) and (5.2),

$$\hat{T}\hat{S} = \begin{bmatrix} \hat{S}_4 & -\hat{S}_2 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} \hat{S}_1 & \hat{S}_2 \\ \hat{S}_3 & \hat{S}_4 \end{bmatrix} = \begin{bmatrix} \hat{S}_4\hat{S}_1 - \hat{S}_2\hat{S}_3 & \hat{S}_4\hat{S}_2 - \hat{S}_2\hat{S}_4 \\ \hat{S}_3 & \hat{S}_4 \end{bmatrix}$$

Since $\hat{S}_4$ and $\hat{S}_2$ are lower triangular Toeplitz matrices and therefore commute, we have $\hat{S}_4\hat{S}_2 - \hat{S}_2\hat{S}_4 = 0$. The argument following (3.7) is therefore unnecessary, and we obtain the following theorem.

THEOREM 5.1. *A matrix* $H = [h_{ij}]$, $i, j = 1, 2, \dots, n$, *of order* n *whose* $(1,1)$ *element is nonzero is the inverse of a Toeplitz matrix if and only if there exist polynomials*

$$A(x) = \sum_{\mu=0}^{n-1} \alpha_\mu x^\mu, \quad B(x) = \sum_{\mu=0}^{n-1} \beta_\mu x^\mu, \qquad \alpha_0 \neq 0, \quad \beta = 1, \tag{5.3}$$

where h_{ij} is the coefficient of x^j in

$$x^i\left[A(x)\sum_{\mu=0}^{i-1}\beta_\mu x^{-\mu}-B\left(\frac{1}{x}\right)\sum_{\mu=n-i+1}^{n-1}\alpha_\mu x^\mu\right] \tag{5.4}$$

and the matrix $\hat{S}$ as defined in (5.1) *is nonsingular.*

REMARK 2. It is well known (e.g. [1]) that the Sylvester matrix $\hat{S}$ is nonsingular if and only if $A(x)$ and $x^{n-1}B(1/x)$ are relatively prime. Thus the statement involving $\hat{S}$ could be removed from Theorem 5.1, and replaced by the condition that $A(x)$ and $x^{n-1}B(1/x)$ are relatively prime.

REMARK 3. The coefficients of $A(x)$ and $B(x)$ are precisely the elements which define (2.8) [$\equiv$ (2.9)] in the Toeplitz case, which becomes

$$H=H_1=\begin{bmatrix}\beta_0 & & & \\ \beta_1 & \beta_0 & \mathbf{0} & \\ \vdots & \ddots & \ddots & \\ \beta_{n-1} & \cdots & \beta_1 & \beta_0\end{bmatrix}\begin{bmatrix}\alpha_0 & \alpha_1 & \cdots & \alpha_{n-1}\\ & \alpha_0 & \cdots & \alpha_{n-2}\\ & \mathbf{0} & \ddots & \vdots\\ & & & \alpha_0\end{bmatrix}$$

$$-\begin{bmatrix}0 & & & \\ \alpha_{n-1} & 0 & \mathbf{0} & \\ \vdots & & \ddots & \\ \alpha_1 & \cdots & \alpha_{n-1} & 0\end{bmatrix}\begin{bmatrix}0 & \beta_{n-1} & \cdot & \cdot & \cdot & \beta_1\\ & \cdot & & & & \cdot\\ & & \cdot & & & \cdot\\ & & & \cdot & & \cdot\\ & \mathbf{0} & & & \cdot & \beta_{n-1}\\ & & & & & 0\end{bmatrix}. \tag{5.5}$$

Thus in the Toeplitz case Lemma 2.4 is identical to Theorem 6.1 in [3], and Theorem 5.1 is therefore another way of expressing Theorem 6.3 in [3].

The connection between certain polynomials and the Gohberg-Fel'dman formulae was also discussed in [6].

To illustrate Theorem 5.1, let H be of order 4, and in (5.3) let

$$A(x)=-2-x+2x^2+x^3,\qquad B(x)=1-4x-3x^2+x^3.$$

Since $A(x)=0$ has integer roots, it is easy to check that $A(x)$ and $x^3B(1/x)$ are relatively prime, and so for $\hat{S}$ in (5.1), we have $|\hat{S}|\neq 0$, using Remark 2. Equation (5.5) then gives

$$H=\begin{bmatrix} -2 & -1 & 2 & 1 \\ 8 & 1 & -6 & 2 \\ 6 & 9 & 1 & -1 \\ -2 & 6 & 8 & -2 \end{bmatrix}.$$

Note that, because H is persymmetric, it is only necessary to find h_{ij} for $i+j\leqslant n+1$ ($n=4$ in this example).

To determine A, the inverse of H, we use (3.12) with the submatrices of S replaced by those of $\hat{S}$, and omit the fifth equation in (3.12). This produces the Toeplitz matrix

$$A=\frac{1}{170}\begin{bmatrix} -66 & 84 & -76 & 89 \\ 64 & -66 & 84 & -76 \\ -26 & 64 & -66 & 84 \\ 154 & -26 & 64 & -66 \end{bmatrix}.$$

It is easily verified that $HA=I$.

5.2. *Banded Matrices*

If we now consider H to be a banded matrix, we can reduce the degrees of some or all of the polynomials in Theorems 3.1 and 5.1.

DEFINITION 5.1. If $H=[h_{ij}]$ is an nth order matrix such that $h_{ij}=0$ when $j-i>r$ and $i-j>s$ for some integers r,s, $0\leqslant r,s\leqslant n-1$, then H is said to be a banded matrix. In other words, H is banded if it has triangles of zeros in the upper right and lower left corners. Either or both of these triangles may in fact disappear if r or/and s are equal to $n-1$. If Theorem 3.1 holds, then H can be written in the form (2.8) and so

$$h_{1j}=c^{j-1}(\alpha_{j-1}),\qquad h_{i1}=\alpha_0c^{i-1}(\delta_{i-1}). \tag{5.6}$$

If $H_1=c^{n+1}(JH^TJ)$, it can be written in the form (2.9) by Lemma 2.4 and we also have

$$[H_1]_{1j}=c^{j-1}(\gamma_{j-1}),\qquad [H_1]_{i1}=\gamma_0c^{i-1}(\beta_{i-1}) \tag{5.7}$$

If H is banded with given values of r and s, then it is easy to see that H_1 is also banded with the same r and s. From (5.6) and (5.7) we therefore obtain in this case

$$\alpha_\mu = 0, \quad \mu > r, \qquad \delta_\mu = 0, \quad \mu > s,$$
$$\gamma_\mu = 0, \quad \mu > r, \qquad \beta_\mu = 0, \quad \mu > s,$$

since $\alpha_0 \neq 0$ and $\gamma_0 \neq 0$.

If we assume that $h_{ij} \neq 0$ when $j - i = r$ and $i - j = s$, then $A(x)$ and $C(x)$ are of degree r and $B(x)$ and $D(x)$ are of degree s. Theorem 3.1 can therefore be modified as follows.

THEOREM 5.2. *A matrix* $H = [h_{ij}]$, $i, j = 1,2,\dots,n$, *of order* n, *whose* $(1,1)$ *element is nonzero, is banded so that* $h_{ij} = 0$ *when* $j - i > r$ *and* $i - j > s$ *and* $h_{ij} \neq 0$ *when* $j - i = r$ *and* $i - j = s$, *and is the inverse of a CT matrix, if and only if there exist polynomials*

$$A(x) = \sum_{\mu=0}^{r} \alpha_\mu x^\mu, \quad B(x) = \sum_{\mu=0}^{s} \beta_\mu x^\mu, \qquad \alpha_0, \alpha_r, \beta_s \neq 0, \quad \beta_0 = 1, \tag{5.8}$$

such that h_{ij} *is the coefficient of* x^j *in*

$$x^i \Bigg\{ c^{j-1}[A(x)]\, c^{i-1}\left[\sum_{\mu=0}^{\min(i-1,s)} \delta_\mu x^{-\mu}\right] - c^{j-1}\left[B\left(\frac{1}{x}\right)\right] c^{i-1}\left[\sum_{\mu=\min(n-i+1,r+1)}^{r} \gamma_\mu x^\mu\right]\Bigg\}, \tag{5.9}$$

and S *as defined in* (2.2) *is nonsingular.*

REMARK 4. The coefficients γ_μ, $\mu = 0,1,\dots,r$, and δ_μ, $\mu = 1,2,\dots,s$, $\delta_0 = 1$ are the unique solutions of

$$\sum_{k=0}^{\min(r,s-i+1)} \left[c^{i-1}(\alpha_k)\delta_{k+i-1} - c^{i-1}(\beta_{k+i-1})\gamma_k\right] = 0, \qquad i = 2,3,\dots,s+1, \tag{5.10}$$

$$\sum_{k=0}^{\min(s,r-i+1)} \left[c^{i-1}(\alpha_{k+i-1})\delta_k - c^{i-1}(\beta_k)\gamma_{k+i-1}\right] = 0, \qquad i = 1,2,\dots,r+1. \tag{5.11}$$

Also, in order to check whether S in (2.2) is nonsingular it is only necessary to consider the determinant of a matrix S_1 of order $r+s$ where

$$S_1=\begin{array}{c}\left[\begin{array}{cccccccc}
c^{n-s}(\alpha_0) & \cdots & c^{r+1}(\alpha_r) & \cdot & & & & \mathbf{0} \\
 & \ddots & & & \ddots & & & \\
\mathbf{0} & & & c^{n-1}(\alpha_0) & c^{n}(\alpha_1) & \cdots & & c^{n+r-1}(\alpha_r) \\
\hline
c^{n-s}(\beta_s) & \cdots & & c^{n-1}(\beta_1) & c^{n}(\beta_0) & & & \mathbf{0} \\
 & \ddots & & & & \ddots & & \\
\mathbf{0} & & c^{n+r-s-1}(\beta_s) & \cdot & & \cdots & & c^{n+r-1}(\beta_0)
\end{array}\right]\end{array}\begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} s \\ \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} r\end{array} \tag{5.12}$$

On setting $\alpha_\mu=0$, $\mu=r+1,\ldots,n-1$ and $\beta_\mu=0$, $\mu=s+1,\ldots,n-1$, it is straightforward to show that $|S|$ is nonzero if and only if $|S_1|$ is nonzero.

The following example illustrates Theorem 5.2. Arbitrarily select

$$A(x)=(1+i)+(2-i)x+2x^2,$$

$$B(x)=1+(1-i)x+(3+i)x^2-x^3,$$

and consider matrices A and H of order 5. Then the matrix (5.12) has determinant

$$|S_1|=\begin{vmatrix} 1+i & 2+i & 2 & 0 & 0 \\ 0 & 1-i & 2-i & 2 & 0 \\ 0 & 0 & 1+i & 2+i & 2 \\ -1 & 3-i & 1-i & 1 & 0 \\ 0 & -1 & 3+i & 1+i & 1 \end{vmatrix}=58+27i,$$

showing that the condition on S in Theorem 5.2 is satisfied. From (2.6) we obtain $\gamma_3=\gamma_4=\delta_4=0$ and with $\delta_0=1$,

$$\gamma_0=\frac{5767+497i}{4093},\qquad \gamma_1=\frac{10502+937i}{4093}$$

$$\delta_1=\frac{5251+2515i}{4093},\qquad \delta_2=\frac{12327-2377i}{4093},\qquad \delta_3=\frac{-2635-3132i}{4093}$$

Using (2.8) or (3.1) and $r = 2$, $s = 3$ gives

$$H = \frac{1}{4093}\begin{bmatrix} 4093+4093i & 8186+4093i & 8186 & 0 & 0 \\ 7766+2736i & 17110-3872i & 18688-9123i & 8186 & 0 \\ 14704+9950i & 34797+4837i & 41764-\ \ 882i & 18688+9123i & 8186 \\ -5767+\ \ 497i & 6302-6321i & 37713+1427i & 17206+9068i & 10502-937i \\ 0 & -5767-\ \ 497i & 16804+7258i & 5270+6264i & 5767+497i \end{bmatrix}$$

Finally from (3.12), omitting the sixth equation, we determine the CT matrix

$$A = \frac{1}{8186}\begin{bmatrix} -1484-\ \ 156i & 526+\ \ 386i & 1044-486i & -2721-\ \ 67i & 3490+\ \ 69i \\ 1492-\ \ 130i & -1484+\ \ 156i & 526-386i & 1044+486i & -2721+\ \ 67i \\ 3200+\ \ 204i & 1492+\ \ 130i & -1484-156i & 526+386i & 1044-486i \\ -9234+4722i & 3200-\ \ 204i & 1492-130i & -1484+156i & 526-386i \\ 6052+\ \ 570i & -9234-4722i & 3200+204i & 1492+130i & -1484-156i \end{bmatrix},$$

and it can be checked that $HA = I$ as required.

REMARK 5. The equivalent of Theorem 5.2 for Toeplitz matrices will have $A(x) = C(x)$, $B(x) = D(x)$, so (5.10) and (5.11) are not needed and (5.9) is replaced by

$$x^i\left[A(x)\sum_{\mu=0}^{\min(i-1,s)}\beta_\mu x^{-\mu} - B\left(\frac{1}{x}\right)\sum_{\mu=\min(n-i+1,r+1)}^{r}\alpha_\mu x^\mu\right]. \quad (5.13)$$

If we now further restrict the banded matrix H so that $r+s<n$, we can consider h_{ij} separately for different values of i. Firstly, if $i \leqslant n-r$, then $\sum_{\mu=\min(n-i+1,r+1)}^{r} = 0$ and so (5.9) reduces to

$$x^i c^{j-1}[A(x)]c^{i-1}\left[\sum_{\mu=0}^{i-1}\delta_\mu x^{-\mu}\right], \qquad 1 \leqslant i \leqslant s, \quad (5.14)$$

$$x^i c^{j-1}[A(x)]c^{i-1}\left[D\left(\frac{1}{x}\right)\right], \qquad s+1 \leqslant i \leqslant n-r. \quad (5.15)$$

Secondly, if $i > n-r$, (5.9) becomes

$$x^i\left\{c^{j-1}[A(x)]c^{i-1}\left[D\left(\frac{1}{x}\right)\right] - c^{j-1}\left[B\left(\frac{1}{x}\right)\right]c^{i-1}\left[\sum_{\mu=n-i+1}^{r}\gamma_\mu x^\mu\right]\right\},$$

$$n-r+1 \leqslant i \leqslant n. \quad (5.16)$$

The restriction to $r+s<n$ also gives the following result.

THEOREM 5.3. *If $H=[h_{ij}]$, $i,j=1,2,\dots,n$, is the inverse of the CT matrix $A=[a_{ij}]$ of order n, where $a_{ij}=c^{i-1}[a_{i-j}]$, and is banded according to Definition 5.1 with $r+s<n$, then $h_{11}\neq 0$.*

Proof. Since $H=A^{-1}$, we have $HA=AH=I$, which gives

$$\sum_{\nu=1}^{r+1} c^{\nu-1}(a_{\nu-j})h_{1\nu}=\delta_{1j}, \qquad j=1,2,\dots,n, \tag{5.17}$$

and

$$\sum_{\mu=1}^{s+1} c^{i-1}(a_{i-\mu})h_{\mu 1}=\delta_{i1}, \qquad i=1,2,\dots,n. \tag{5.18}$$

Let p be the smallest integer such that $h_{\mu 1}\neq 0$ (note that $1\leqslant p\leqslant s+1$), and consider

$$\Lambda=\sum_{\mu=1}^{s+1} c^{p-1}(h_{\mu 1})\sum_{\nu=1}^{r+1} c^{\nu-1}(a_{p+\nu-\mu-1})h_{1\nu}. \tag{5.19}$$

Now $h_{\mu 1}=0$ for $\mu<p$, and using (5.17) when $\mu\geqslant p$ shows that

$$\Lambda=c^{p-1}(h_{p1}) \tag{5.20}$$

Reversing the order of summation in (5.19) and using (5.18) gives

$$\Lambda=\sum_{\nu=1}^{r+1} h_{1\nu}\sum_{\mu=1}^{s+1} c^{\nu-1}(a_{p+\nu-\mu-1})c^{p-1}(h_{\mu 1})$$

$$=\sum_{\nu=1}^{r+1} h_{1\nu}c^{p-1}(\delta_{\nu+p-1,1}). \tag{5.21}$$

If $p>1$, then (5.21) shows that $\Lambda=0$, contradicting (5.20), since $h_{p1}\neq 0$. If $p=1$, (5.21) agrees with (5.20), and the result is established. ■

Note that in order to obtain (5.21) using (5.18) it is necessary to have $\nu+p-1\leqslant n$, $\nu=1,2,\dots,r+1$. Since $p\leqslant s+1$, this shows that $r+s$ must be less than n.

REMARK 6. If Theorem 5.2 is restricted to the Toeplitz case together with $r + s < n$, then (5.9), which has been replaced by (5.14), (5.15), and (5.16), is further reduced to

$$x^i A(x) \sum_{\mu=0}^{i-1} \beta_\mu x^{-\mu}, \qquad 1 \leqslant i \leqslant s, \tag{5.22}$$

$$x^i A(x) B\left(\frac{1}{x}\right), \qquad s+1 \leqslant i \leqslant n-r, \tag{5.23}$$

$$x^i B\left(\frac{1}{x}\right) \sum_{\mu=0}^{n-i} \alpha_\mu x^\mu, \qquad n-r+1 \leqslant i \leqslant n. \tag{5.24}$$

Hence, when $r + s < n$, Theorem 5.2 and Theorem 5.3 together reduce in the Toeplitz case to the main theorem in [5]. It does not seem to have been realized in [5] that this theorem is a different way of expressing a result in [3], restricted to the banded case $r + s < n$, or that this restriction can be removed as shown in Theorem 5.1. It should also be pointed out that the theorem in [5] is not stated in the strongest form. It should be worded as in Theorem 5.2, since the banded form of H is a consequence of (5.9), and not a prerequisite.

The authors wish to thank the referee for helpful comments and suggestions.

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Received 23 December 1982; revised 18 July 1983